A Proof of the Collatz Conjecture by Synthesis

Version: 6.7 Author: Franz Ziegler

Date: Dec. 10. 2020 ORCID 0000-0002-6289-7306

1. Summary

All natural odd numbers are contained in the infinite Collatz tree and are defined as members of a LN terminated chain module too. The proposed definition and configuration of such a module has turned out to be an valuable enclosure for the synthesis of the building blocks of the Collatz tree. A logical chain of conditions was used to adapt the Collatz rules and customizing the definition range of the operations in such a way, that the Collatz rules can be uses like bijective functions. The paper not only confirms the Collatz Conjecture but also gives insight into some important key points of the Collatz number structure.

2. Prerequisite

We assume the standard collatz number series and the resulting collatz number tree.

3. Methods

An encapsulation module called LN terminated Chain Module is defined containing slightly modified Collatz rules with additional stop condition. It's configuration empowers the Collatz rules to be used as bijective functions. Together with the one-sided connection capability of LN chains, loops and holes will be awoided with later assembly. The synthesis will start by filling in the numbers into the LN chains. The following assembly connects any LN chains together and contains all natural numbers exactly once. The assembled structure is identical with the infinite Collatz tree.

4.1 Definition of the LN terminated Chain Module

An LN terminated chain module or simply a LN chain is a limited series of natural numbers with one-sided connection capabilities to link with other chains (LN = Link – Nolink). Every number has a predecessor and a successor inside or outside of the module. The sequence of numbers is subject of rules. Internal rules can differ from global ones. The internal operations defined to propagate the number series have to be diophantine bijective functions [02] with [f(x) <> x]. Additional internal rules for shortening the definition range of functions can help to achieve this goal. The -N terminal has NO connection capability to other LN chains, but connections to even numbers are possible. The L-terminal must have LINK capability to connect with members of other LN chains. The L- terminal of the chain will also be called "foot" and the N terminal will be called "head".

The definition of LN chains is motivated by the fact, that in Collatz sequencies, every third odd number has only even numbers as predecessors. so there is an infinite number of branches above such odd numbers having no connection to other branches. Such odd "leafs" are used as starting point and as -N terminators to generate the number sequences inside the chains.

4.2 Configuration of the Module Using modified Collatz Rules

The Collatz rules

$g = (3u+1)$ u: { 1, 3, 5, 7, 9, ...} g:{ 2, 4, 6, 8, ..} (1)

as well as

$x = g / (2^n)$ g: {2, 4, 6, 8, ...} n: {1, 2, 3, 4, ...} (2)

will be used. The function (2) is used in the same way as in the standard Collatz series: divide by two until an odd number is reached, but the repetition of the division is limited to two times per round, so that the LN limited chain will have one or two even numbers maximum between odd numbers.

This limitation is performed with the filter or stop condition:

$(u - 5) \bmod 8 = 0;$ u: {1, 2, 3, 4, ...}

The stop condition will limit the chain lenghts, awoid nodes and has the same function as cutting the branches along the proxy elements where the the proxy is the first daughter of a generation [01].

5.1 Applying the Module for the Synthesis of Collatz Buildung Blocks

For creating the Collatz tree, every third odd natural number can be uses als -N or head terminal of an LN module because these Collatz numbers have no odd predecessor:

no odd predecessors: u mod 3 u: {1, 3, 5, 7, 9, 11, ...} (1)

This can be shown by combining the condition (1) with the Collatz rules. The following equation contains the Collatz rules at the left side and the condition (1) at the right side. The uneven equation is true for all combinations of natural numbers n,m,k:

$(3*(2n-1)+1) / 2^k <> (2m-1) / 3$ {n,m,k: 1, 2, 3, 4, ...}

For that the following odd numbers are all heads of an LN chain:

heads: $h = 3*(2*n+1) =$ { 3, 9, 15, 21, 27, ...} {n: 0, 1, 2, 3, ...}

The LN chains are now filled with limited Collatz series. The propagation runs from head to foot, and using the Collatz rules:

Op1: $z = (z * 3+1) / 2$ {z: 1, 3, 5, 7, 9, ...} (2)

(Op2: only if z = even) $z = z / 2$ (3)

If the Collatz conjecture is true, the Op1 .. Op2 operation should not yield to a loop. The two cases have to be tested:

a) z is even after OP1: $z = (3z+1) / 2$ {z: 1, 3, 5, 7, ...}

$z = -1$ no loop: out of definition range

b) z is odd after OP1: z = (3z+1) / 4

z = 1 loop !

There is a unallowed loop in the Operation. But because this is the known loop at the end of any Collatz number series, we don't care about that and will not discuss it further.

We will get always an odd number after using Op1 or Op1...Op2. Now we apply the additional stop condition of the LN chain:

rule: exit if: (z – 5) mod 8 = 0 else repeat Op1, (Op2)

It it exits, the LN chain is completed. The last number we got is the foot termination of the chain.

If the Operations gets repeated we have to ensure that also in this case no loop will occur. Unlike the Collatz rule, whereas the division /2 is applied on even numbers until an odd number is reached, the Op1, Op2 are limited by only two division /2. With that limitation, only one or two even numbers can occur maximum between two odd numbers. This makes (Op1 .. Op2) a propagating function unable to create loops. One ore two divisions are allowed, because only every second even number is connected to an odd number. By jumping from one generation to the next, this can cause a shift of one division (see [01]). In combination with the terminating head as starting point without any odd number predecessor, loops can not occur. The internal number series in the LN chain will propagate until the stop condition ((z – 5) mod 8 = 0) is reached marking the foot

The proposed operations will lead to a complete set of all possible LN chains flowing in the universe now. This set is complete, because as defined, all odd numbers are contained in exactly one LN chain. The numbers in the chains also do not overlap, because every chain has it's unique head: h: {3, 9, 15, 21, 27, 33, ...}

5.2 Assembling the Collatz Tree

With the successfully completed chains the proof of the Collatz conjecture is reachable now, because according the definition of LN chains one part of the remaining even numbers will be used to continue at the H- (head) termination where they will not have connection capabilities. The other part of unused even numbers will be used at the F- (foot) termination to form the links between LN chains.

Nevertheless the universe of floating LN chains will be assembled now to the Collatz tree by applying two additional steps. The first step will join the LN chains together. For that, the standard Collatz series is continued at the foot with three additional steps using the Collatz rules. With that, the third is only created to hold the join information and you will find the same number within a chain created before. Be aware that the join elements can be found in the whole structure produced yet. The links links will never be found upward the head. The continued Collatz series, started at the foot element 53 now looks like this:

(40) 80 160 **53** 106 35 70 23 46 **15**

The number (in parantheses) now is no more part of the joining chain, but belongs to the chain to be linked to. With the second step we will complete the head side of the chain with the missing 2^n number series: the LN chain with head h = 15:

(40) 80 160 **53** 106 35 70 23 46 **15** (= head)

the series h*2^n elements will be added:

(40) 80 160 **53** 106 35 70 23 46 **15** 15*(2^1) 15*(2^2) 15*(2^3) 15*(2^4) ...

= **(40)** 80 160 **53** 106 35 70 23 46 **15** 30 60 120 240 480 960 ...

The completed chain can now be connected to other completed chains:

160 **53** 106 35 70 23 46 **15** 30 60 120 ...
80
40 **13** 26 52 17 34 11 22 7 14 28 **9** **1**8 36 72 ...
20
1 2 4 8 16 **5** 10 **3** 6 12 24 ...

With the completion of the number series at both ends of the LN chains and the assembly of all the infinite LN chains the synthesis of the infinite tree is done. The Collatz conjecture has been confirmed as true at this point as the resulting number structure corresponds exactly to the Collatz tree.

6. Literature

[01] FRANZ ZIEGLER: Proof of the collatz conjecture V1.2 www.collatz.com

[02] HORST STÖCKER: taschenbuch mathematischer Formeln und moderne Verfahren. Verlag Harry Deutsch (ISBN 3-8171-1572-5)